Justifying geometrical generalizations in elementary school preservice teacher education

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This paper reports research based on a K-6 prospective teacher education experiment carried out in a geometry course in the 2nd year of studies. The study aims to understand how participants justify generalizations about families of geometric figures in a context of exploratory teaching. Data were collected by audio and video records and from participants' written productions. In the analysis, special attention was given to the kind of arguments, their degree of generality, and the aspects that contribute for the learning of the justification process. The results show that initially the participants had difficulties in understanding how to justify generalizations. They progressed by using valid arguments, but they struggled in fully providing arguments and reasoning beyond specific cases. An improvement of justifications was achieved by the careful design of tasks, the interaction in the classroom and by relating the process of justification to understanding why a statement is true.

Keywords: Geometry, Reasoning, Justification, Generalization, Preservice Teachers Education.

Introduction

Lo and McCrory (2009) argue that prospective elementary teachers need to learn proof: a) as a tool to show or verify that something is true or false; b) as a mathematical object that is regulated by some rules and standards; and c) as a factor of students' development. These levels correspond to knowing how to proof, understand the nature of proof and to adapt proof to different students' developmental levels. However, Stylianides and Stylianides (2009) refer several studies showing that prospective elementary teachers have predominantly misconceptions about proof, particularly regarding the role of empirical arguments. Also Lin et al. (2012) add that, for many of these teachers, their belief in a result rests more on the authority of external entities than on their reasoning.

For Stylianides, Bieda and Morselli (2016), in the last decade, some research studies sought ways to support students in argumentation and proof, particularly in geometry. In teacher education, according to Lin et al. (2012), some studies suggest guidelines to improve the knowledge of prospective teachers in proof: solve tasks individually or in small groups; hold collective discussions; share and criticize one another's proofs; promote cognitive challenges. In geometry, the use of DGS and of suitable tasks may motivate the search for justifications to explain why conjectures are true (Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004). However, as Stylianides et al. (2016) suggest, in this area, there is still a need for research in designed interventions that focus on the development of prospective teachers' mathematical knowledge about proof. Assuming this, our paper addresses the need to support future teachers in the process of justification in geometry. Its purpose is to understand how they justify generalizations about families of geometric figures. We analyse the following questions: what kind of arguments do participants use to justify generalizations about families of geometric figures? What are the obstacles and the facilitating aspects of learning to justify suggested by the experience?

Mathematical reasoning and the process of justification

Lannin, Ellis e Elliot (2011) consider mathematical reasoning as an evolving process of conjecturing, generalizing, investigating *why*, justifying and refuting assertions. For these authors, generalizing is about identifying common elements or extending the reasoning beyond the range in which it originated. Investigating *why* involves investigating factors that may explain why a generalization is true or false. A valid justification constitutes a logical sequence of statements, each relying on established knowledge, in order to arrive at a conclusion; it must use general language demonstrating that it applies to more than one particular case, even if it is based on generic examples. In the context of teaching, a successful justification shows that a statement is true and explains *why* it is true.

Considering this characterization, we find the concepts of justification and proof to be very close, which results from the several meanings attributed to proof, both in research in mathematics education (Stylianides et al., 2016) and in mathematics, where there are many conflicting opinions about the role of proof and what makes a proof acceptable (Hanna, 2000; Harel & Sowder, 2007). Stylianides (2007) presents a definition based on the literature on the philosophy of mathematics and mathematics education that addresses mathematics teaching from the first years of schooling:

> A proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known, or within the conceptual reach, of the classroom community. (p. 291)

Considering that we focus on justifying[1] generalizations concerning geometrical objects, the statements must relate to the geometrical structure of the objects. In this sense, we call on the ideas of Battista (2009), suggesting that reasoning involves spatial structuring—a special type of abstraction corresponding to the mental act of constructing an organization or form for an object or set of objects by identifying its components, combining them into spatial composites, and identifying the way they combine and relate—and geometric structuring (GS), which describes spatial structuring using formal concepts. Also, we should also consider Balacheff's (1988) "generic example" as a form of reasoning particularly suitable for justifying geometrical generalizations, as it "involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there by its own right, but as a characteristic representative of its class" (p. 219).

Methodology

This paper addresses an investigation with an intervention, in order to change practices and enhance teachers' preparation in geometry. The research focus is on learning in context, starting from the

[1] From now on, instead of "proof", we use the term "justification".

conception of strategies and teaching tools, following a design-based research as methodology, in the form of a prospective teacher experiment (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) in which the teacher also plays the role of researcher. This approach is referred by Stylianides et al. (2016) as “a promising approach to respond to the need for developing effective ways to address students’ and teachers’ difficulties with argumentation and proof” (p. 344). Two of the design principles of the experience influence directly the tasks that we report in this paper: (i) make use of the intimate relation between sense making and the activity of reasoning and proving to promote learning with understanding; (ii) promote flexible reasoning, providing tools for prospective teachers, including different ways of justifying (Stylianides & Stylianides, 2006).

The data were collected during the second cycle of the study, involving a group of 25 trainees who attended a Geometry course (2^{nd} year of the Basic Education Bachelor’s Degree). The tasks were solved in groups of 4/5 elements. Data gathered includes the participants’ records of two tasks solved in the classroom and audio and video recordings of the groups’ interaction.

We present a framework to analyse the justification of generalizations (Table 1) taking in account, first, the nature of the arguments regarding the geometric structuring of objects which relate mainly to the properties stated and, second, the degree of generalization of the justification.

Level	Argument’s nature	Properties / procedures	Degree of generalization
GS3	Based on the correct geometric structuring of the family of figures	States relevant and established properties	Uses a generic language about the family of figures
			It focuses on a generic example
			It focuses on one or more figures without generalizing
GS2	Based on the incomplete geometric structuring of the family of figures	States relevant and established properties, but omits others	Uses a generic language about the family of figures
			It focuses on a generic example
			It focuses on one or more figures without generalizing
GS1	Based on the incorrect geometric structuring of the family of figures	States irrelevant, non-existent or non-established properties	Uses a generic language about the family of figures
			It focuses on a generic example
			It focuses on one or more figures without generalizing
GS0	Without resorting to the geometric structuring of the family of figures	States numerical relations without connection to the structuring of the figures	It focuses on one or more figures
		Tests the generalization	It focuses on one or more figures
		Uses an external source of validation	Does not apply

Table 1: Levels for justifications of generalizations about families of figures

Results and discussion

In this section, we discuss results from two tasks, one about the congruence of the vertically opposite angles and another about the sum of the amplitudes of the internal angles of a polygon. In a previous lesson, the participants used GeoGebra to conjecture about these relationships, but they were not supposed to use it in these tasks.

Task A – Vertically opposite angles

Previously you discovered that two vertically opposite angles have the same amplitude. Find a justification explaining why this relationship is always true.

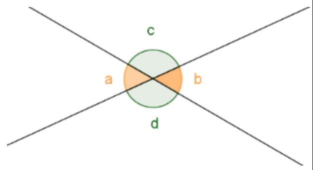

Figure 1: Task for the justification of the congruence of vertically opposite angles

This task was not the first asking for a justification involving angles, but it was the first one using a generalization in which no value was given, so the reaction of the participants was very different from the previous ones. There were only two written answers, one of them from Helena (Figure 2):

As the vertically opposite angles have the same vertices (the straight lines always pass through the same point) any way we put the straight lines (any orientation), the angles were always equal.

Figure 2: Helena's justification for task A

Helena's answer refers to a characteristic of vertically opposing angles, but her justification does not resort to geometric structuring because, by stating "any way we put the straight lines," she is drawing on her prior experience with GeoGebra. In this way, her justification implicitly refers to an external source to validate the claim, so the justification is incorrect (level GS0). Although based on empirical experience, the software represents the authority in which Helena trusts.

The other written answer is similar to most reactions, illustrated by the following dialogue:

Marina: They have to be equal because they have the vertex in common and the sides of one angle are the sides of the other.

Teacher: But what you are saying to me is almost the definition of vertically opposite angles. This statement does not justify the claim.

Marina: So how do we justify it?

In an attempt to help the group, the teacher suggests introducing a value:

Teacher: Imagine that *a* is equal to 30º. Try to find the values of the other angles without using the property.

Marina: Which property?

Teacher: The one that you want to justify. That vertically opposing angles are congruent. Find the other values from other relationships.

Marina: Oh! So... *c* is 150... because adding *a* it gives 180 degrees. They are... supplementary.

Teacher: OK...

Marina: Then *b* is 30 because it is vertically opposite to *a*.

Teacher: Attention! We agreed that we can not use this property. Do you understand why? You cannot justify that a property a true if you are using it in your reasoning.

Marina: OK... Hum, *b* is 30 because it's supplementary to *c*, which is 150.

Teacher: OK. As you can see you discovered the values 30 and 150 without using the property. Now, try to use a similar reasoning without using a specific value.

Marina shows some difficulties. On the one hand, she does not distinguish the characterization of the vertically opposing angles from the justification of their congruence. This problem may be due to the strong perception that angles have to be congruent by the way they are constructed. On the other hand, the simplified version of the problem using a specific value also shows that Marina does not know she cannot use the property she is seeking to. The group made an effort to continue the task, but they struggled to generalize the justification. Thus, this episode shows an answer based on an incomplete geometric structuring of the family of figures (EG2) which focused on a particular figure without generalizing.

Task B – Sum of the amplitudes of the internal angles of a polygon

You have found a generalization for the sum of the internal angles of a polygon using GeoGebra. Let's try to justify it. To do this, look at the following hexagons. Each one suggests a possible strategy. Use one of the strategies to write the justification.

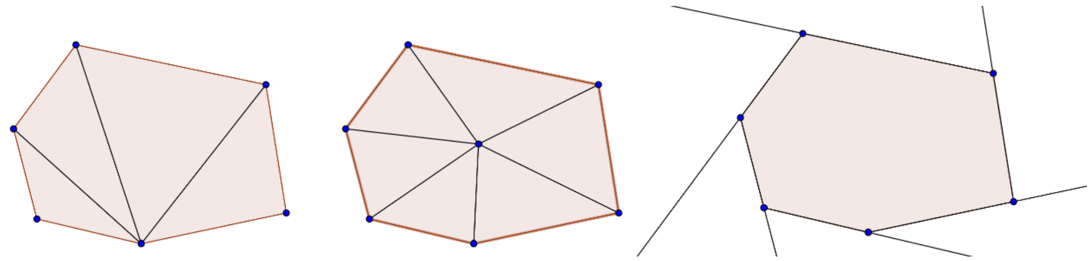

Figure 3: Task for justification of the sum of internal angles of a polygon

In the beginning, the prospective teachers struggled again with the absence of values because some thought that they would need the value of each angle, but the teacher then stressed that they should continue the strategies presented. This time, all the groups were able to produce some justification.

The first hexagon is divided into 4 triangles. All vertices of each triangle cover all the internal angles of the polygon. If we know that the sum of the internal angles of a triangle is equal to 180°, we can multiply 180 by 4 (4 triangles) and we obtain the amplitude of the whole polygon. The expression that generalizes is (n-2)×180. If a polygon has 10 sides, it is possible to draw 8 triangles; if you have 6 sides, we draw 4 triangles. If we have n sides, we draw n-2 triangles.

Figure 4: Celia's justification using the strategy of the first figure

Celia's answer (Figure 4) is based on the correct geometric structuring because it identifies two relevant properties (the sum of the amplitudes of the internal angles of a triangle and the decomposition the polygon into *n-2* triangles). Celia uses the hexagon and the decagon with the intention to treat them like generic examples, because it explicitly indicates properties of the class. In this way, her justification is at level GS3[2].

All groups used the strategy initiated in the first figure, but most of them decided to follow the other strategies as well. The next answer (Figure 5) belongs to Anita:

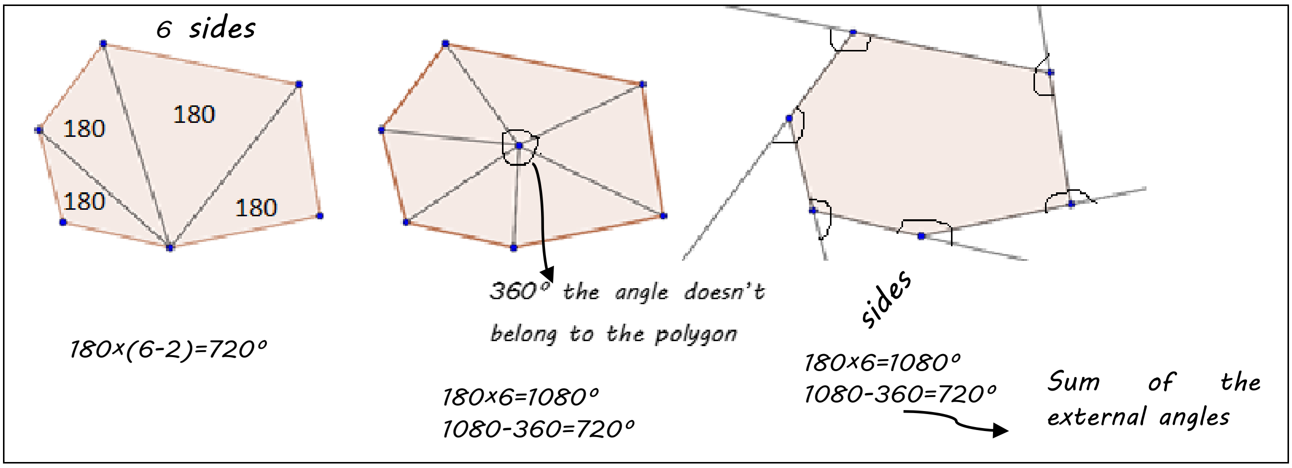

Figure 5: Translated reproduction of Anita's justification

Anita's justification is based on a correct geometric structuring, using relevant and established properties, although she does not explain the relation between the number of sides of the polygon and the number of triangles. In addition, the properties concern only the case of the hexagon, which is not used as a generic example, and are not explained properly, so the justification is incomplete although refers to level GS3. The restriction to the hexagon is a common problem addressed by the teacher:

Teacher: But that's for the hexagon. What about other polygons? For example, a decagon?

Isabel: We use... 8 triangles. We took two sides.

Teacher: So?

Isabel: Exactly. Then it gives (n-2) × 180!

Andreia: And for the other we do 6 × 180 and then we take two triangles. 2 × 180.

Teacher: And why do you take the angles of two triangles?

Isabel: Yeah... This is you forcing to give the same result...

Teacher: That's it. If you have to take something, some value, that has to make sense...

[2] We are only considering the quality of the arguments regardless of language errors.

In this dialogue, we observe that the participants are trying to extend the first strategy to the second case without understanding why. When the teacher confronts them, they recognise their problem.

Conclusion

Task A showed trainees' difficulties related to two factors: the nature of the statement to be justified —a generalization—supported by a generic representation with no values; the principles of a justification, namely the impossibility of using a single example or relying on cyclic reasoning. The fact that the participants are able to solve a similar task by introducing a value shows that difficulties may not arise from the identification of relevant and established properties, but from the construction of an argument that applies this structuring to the entire family of figures. This means that the prospective teachers showed difficulties both in justifying and in understanding the nature of justification (Lo & McCrory, 2009).

The solutions of task B show a correct geometric structuring of the family of figures, using valid arguments, even if incomplete. Most justifications tend to be supported by specific examples, but in some cases, the participants try to present them as generic. The teacher suggests that the justifications shows why the relation is true and participants seem to accept that suggestion.

Thus, the two tasks show some differences with respect to the type of arguments used by the participants to justify generalizations, since they started to rely more on the correct structuring of the geometric figures. These differences may derive from the specificity of tasks, but may also correspond to a more correct conception of what a justification means. However, the solutions from task B show that there are two important aspects to attend. On the one hand, it is necessary to overcome the resistance in constructing an argumentative discourse, which we observe in justifications that are reduced to the schematic interpretation of expressions or visual representations, in order to value the communicative dimension of this process (Yackel & Hanna, 2003). On the other hand, it is important to raise the degree of generality of the discourse which, in some cases, is overly supported by particular examples and does not show that generalization applies to the whole domain of figures (Lannin et al., 2011). In fact, there is an unclear line between presenting a generic example that is representative of the domain (an acceptable strategy to justify) and supporting a justification by empirical examples, which corresponds to a common error and a misconception about the role of empirical results in the validity of a justification (Stylianides & Stylianides, 2009).

The results presented refer only to two tasks used to promote the ability to justify generalizations. However, they confirm the relevance of relating the process of justification to understanding why a statement is true, suggested by several authors (e.g., Harold & Sowder, 2007; Lannin et al., 2011; Stylianides et al., 2016). In particular, the design of tasks that promote the construction and confrontation of different justifications and representations, as well as an environment of peer interaction, seem to be contributing factors in the development of the ability to justify.

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